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On the countable generator theorem

by

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Abstract. Let T be a finite entropy, aperiodic automorphism of a nonatomic probability space. We give an elementary proof of the existence of a finite entropy, countable generating partition for T.

In this short article we give a simple proof of Rokhlin's countable generator theorem [Ro], originating from considerations in [Se] which use standard techniques in ergodic theory. We hope that these considerations will be useful for elementary expositions in the future. For other proofs see [Pa].

Let $(X, \mathcal{A}, \mu)$ be a nonatomic probability space whose σ-algebra $\mathcal{A}$ is generated modulo μ by a countable collection $\{A_1, A_2, \ldots\}$ of elements of $\mathcal{A}$. Let T be an aperiodic automorphism of $(X, \mathcal{A}, \mu)$ with finite entropy. For the definitions and properties of entropy and generators used in the sequel, we refer the reader to Billingsley [Bill] and Walters [Wa].

THEOREM. *$(X, \mathcal{A}, \mu, T)$ has a countable generating partition of finite entropy.*

Our proof is based on the following lemma.

LEMMA. *Let $\mathcal{P}$ be a finite partition of $(X, \mathcal{A}, \mu, T)$, A an element of $\mathcal{A}$, and $\varepsilon > 0$. Set*

$$\widetilde{\mathcal{P}} := \mathcal{P} \vee \{A, A^c\} \quad \text{and} \quad g := \widetilde{h} - h,$$

where

$$h := h(T, \mathcal{P}) \quad \text{and} \quad \widetilde{h} := h(T, \widetilde{\mathcal{P}})$$

denote the respective mean entropies of the partitions $\mathcal{P}$ and $\widetilde{\mathcal{P}}$. Then there exists a finite partition $\mathcal{Q}$ of $(X, \mathcal{A}, \mu, T)$ such that

$$\mathcal{P} \preceq \mathcal{Q}, \tag{1}$$

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$$(2) \qquad A \in \bigvee_{n=-\infty}^{\infty} T^n \mathcal{Q},$$

$$(3) \qquad H(\mathcal{Q}) \le H(\mathcal{P}) + g + \varepsilon.$$

Assuming the validity of this lemma, here is the proof of the theorem: Using the lemma, we produce inductively a sequence $\mathcal{Q}_0 \preceq \mathcal{Q}_1 \preceq \dots$ of finite partitions as follows. First, set $\mathcal{Q}_0 = \{X\}$. If $\mathcal{Q}_k$ has been defined, then take

$$\varepsilon = \frac{1}{2^{k+1}}, \quad \mathcal{P} = \mathcal{Q}_k, \quad A = A_{k+1}$$

in the lemma to obtain $\mathcal{Q}_{k+1} := \mathcal{Q}$. By (1) and (2), for each $k \ge 0$,

$$A_1, \dots, A_k \in \bigvee_{n=-\infty}^{\infty} T^n \mathcal{Q}_k.$$

Moreover, property (3) yields

$$H(\mathcal{Q}_{k+1}) - H(\mathcal{Q}_k) \le h(T, \mathcal{Q}_{k+1}) - h(T, \mathcal{Q}_k) + \frac{1}{2^{k+1}}$$

for each $k \ge 0$; summing from zero to k results in

$$H(\mathcal{Q}_k) \le h(T, \mathcal{Q}_k) + \sum_{j=1}^{k+1} \frac{1}{2^j} \le (\text{Entropy of } T) + 1.$$

In particular, $\sup_k H(\mathcal{Q}_k) < \infty$. Now set

$$\mathcal{Q} := \bigvee_{k=0}^{\infty} \mathcal{Q}_k;$$

then $H(\mathcal{Q}) = \sup_k H(\mathcal{Q}_k)$ is finite, and $A_k \in \bigvee_{n=-\infty}^{\infty} T^n \mathcal{Q}$ for each k, so that $\mathcal{Q}$ is a countable generating partition of finite entropy. ∎

Next, we give a proof of the lemma in the case where T is ergodic. It is clear that we may replace the condition (2) by the condition

$$(4) \qquad \text{there exists an } A' \in \bigvee_{n=-\infty}^{\infty} T^n \mathcal{Q} \text{ with } \mu(A \,\triangle\, A') < \varepsilon.$$

To see this, suppose that the lemma holds in this modified form, and for $\varepsilon > 0$ choose $\delta > 0$ such that

$$\delta + \{-\delta \log \delta - (1-\delta)\log(1-\delta)\} \le \varepsilon.$$

Apply the modified lemma using δ in place of ε to get a partition $\mathcal{Q}'$ satisfying (1), (4), and (3). Then

$$\mathcal{Q} := \mathcal{Q}' \vee \{A \,\triangle\, A', X \setminus (A \,\triangle\, A')\}$$

satisfies (1) and (2), and

$$\begin{aligned} H(\mathcal{Q}) &\le H(\mathcal{Q}') + H(\{A \triangle A', X \setminus (A \triangle A')\}) \\ &\le H(\mathcal{P}) + g + \delta + \{-\delta \log \delta - (1-\delta)\log(1-\delta)\} \\ &\le H(\mathcal{P}) + g + \varepsilon \end{aligned}$$

as required.

For a fixed positive integer m, which we shall choose in a moment, let

$$\{A_{ij} : 1 \le i \le p^m, 1 \le j \le 2^m\}$$

be a list of the (possibly empty) atoms of $\bigvee_{n=0}^{m-1} T^n\widetilde{\mathcal{P}}$ such that the sets

$$A_i := \bigcup_{j=1}^{2^m} A_{ij}$$

are the atoms (possibly empty) of $\bigvee_{n=0}^{m-1} T^n\mathcal{P}$; here we have assumed that $\mathcal{P}$ has p elements.

By the Shannon–McMillan–Breiman theorem (what we need here is convergence in probability, see [Bill], Thm. 13.2), if $\delta > 0$ and m is large enough, "most" of the A_{ij} have measures in

$$[e^{-(\tilde{h}+\delta)m}, e^{-(\tilde{h}-\delta)m}]$$

and "most" of the A_i have measures in

$$[e^{-(h+\delta)m}, e^{-(h-\delta)m}],$$

"most" meaning, of course, a set with total measure close to 1. For a $\delta > 0$ also to be determined shortly, we now choose m so large that the total measure of the atoms A_i for which

$$\mu(A_i) > e^{-(h-\delta)m} \tag{5}$$

is smaller than δ, and also so that the total measure of the atoms A_{ij} for which

$$\mu(A_{ij}) < e^{-(\tilde{h}+\delta)m} \tag{6}$$

is smaller than δ.

Next, we reorganize the array $\{A_{ij}\}$ as follows. First, delete all the rows i for which (5) holds. Then, in the remaining rows, delete all the A_{ij} for which (6) holds. Finally, renumber the remaining elements to obtain the array

$$\{A'_{ij} : 1 \le i \le I,\ 1 \le j \le J_i\},$$

each row of which is a subcollection of a row of the original array. Since now still

$$\sum_{j=1}^{J_i} \mu(A'_{ij}) \le e^{-(h-\delta)m}$$

for each row i, and since $\mu(A'_{ij}) \geq e^{-(\tilde{h}+\delta)m}$ for each of the remaining elements of a row, it follows that for each $1 \leq i \leq I$,

$$J_i \leq \frac{e^{-(h-\delta)m}}{e^{-(\tilde{h}+\delta)m}} = e^{(g+2\delta)m}.$$

If $\bar{J} := \max_{1\leq i\leq I} J_i$, and $1 \leq j \leq \bar{J}$, then we set

$$Q'_j := \bigcup_{\{i: j \leq J_i\}} A'_{ij}.$$

Now use Rokhlin's lemma to get a set $M \in \mathcal{A}$ such that $M, TM, \ldots$ $\ldots, T^{m-1}M$ are pairwise disjoint and

$$\mu\Big(X \setminus \bigcup_{n=0}^{m-1} T^n\Big) < \delta,$$

and define the partitions

$$\mathcal{Q}' := \Big\{M \cap Q'_1, \ldots, M \cap Q'_{\bar{J}}, X \setminus \bigcup_{j=1}^{\bar{J}} M \cap Q'_j\Big\}$$

and $\mathcal{Q} := \mathcal{P} \vee \mathcal{Q}'$. Without loss of generality, by choosing m sufficiently large and by replacement of M by one of the $T^n M$ with n small with respect to m ($n < m\sqrt{3\delta}$ will do), we may assume that

$$\frac{\mu(M \cap \bigcup_{j=1}^{\bar{J}} Q'_j)}{\mu(M)} > 1 - \sqrt{3\delta}.$$

Then, by construction, $\bigvee_{n=-m}^{m} T^{-n}\mathcal{Q}$ contains a set A' with $\mu(A \bigtriangleup A') \leq \sqrt{3\delta}$, namely the union of all its atoms contained in A.

As $\mu(M) \leq 1/m$ and $\bar{J} \leq e^{(g+2\delta)m}$, we have

$$H(\mathcal{Q}') \leq -\bar{J} \cdot \frac{1}{m\bar{J}} \cdot \log\left(\frac{1}{m\bar{J}}\right) - \frac{m-1}{m} \log \frac{m-1}{m} \leq g + 2\delta + \frac{\log m}{m} + \frac{1}{m},$$

and hence

$$H(\mathcal{Q}) \leq H(\mathcal{P}) + g + 2\delta + \frac{\log m}{m} + \frac{1}{m}.$$

Thus choosing δ such that

$$\max\left\{\sqrt{3\delta}, \frac{\log m}{m} + 2\delta + \frac{1}{m}\right\} < \varepsilon$$

finishes the proof. ∎

Finally, we give a sketch of how this proof can be modified for the non-ergodic case. Suppose, for instance, that μ has two ergodic components, say μ_1 and μ_2, with

$$\mu = \alpha\mu_1 + (1-\alpha)\mu_2.$$

Each μ_i corresponds to entropies $\widetilde{h}_i, h_i$ and $g_i = \widetilde{h}_i - h_i$ as above, $i = 1$ or 2. If we produced $\mathcal{Q}_1'$ and $\mathcal{Q}_2'$ as above and joined them to $\mathcal{P}$, the entropy would be too large, and we need to merge the atoms of $\mathcal{Q}_1'$ and $\mathcal{Q}_2'$. For this, the numbers m_1 and m_2 need to be chosen such that $m_1 g_1 \approx m_2 g_2$; all other considerations remain the same. A similar argument applies for arbitrary nonergodic μ by approximation by a finite number of unions of ergodic components with approximately the same $\widetilde{h}$ and h values. The details are left to the reader.

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